2201. Even functions have graphs with $x = 0$ as a line of symmetry; odd functions have graphs with $(0, 0)$ as a centre of rotational symmetry.

The function $\mathrm{f}(x) = x^3 + 1$ cannot be even, as x^3 is an odd power. And, while the curve $y = x^3$ does have odd symmetry, we have $y = x^3 + 1$, which is translated in the y direction, away from the origin. So, neither can f be odd.

2202. (a) This is clearly false: 2×1 and 3×1 rectangles provide a counterexample.

(b) This is true. Kites have a line of symmetry, which splits them into two triangles. Since the line of symmetry bisects the angles, the two triangles in one kite and the two triangles in the other kite must all have the same angles, so they must be similar. Hence, the kites are necessarily similar.

2203. Factorising,

$$x^2 - 4x + 3 \equiv (x - 1)(x - 3).$$

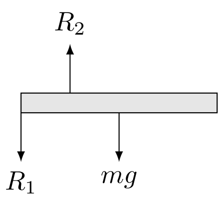
Consider the factors $(x-1)$ and $(x-3)$, using the factor theorem. Evaluating the cubic at $x = 1$ and $x = 3$ gives -3 in both cases. Since, this is non-zero, $(x-1)(x-3)$ cannot be a factor, and so must leave a non-zero remainder.

2204. Any $x \in S$ must be a root of E, because it renders both f and g zero, thus equal. So (b) is true. But (a) and (c) are not. A counterexample to both is

$$\begin{aligned} \mathrm{f}(x) &= x^3 + 2x = 0, \\ \mathrm{g}(x) &= 2x^3 + x = 0. \end{aligned}$$

Each has solution set $\{0\}$, but the solution set of $\mathrm{f}(x) = \mathrm{g}(x)$ is $\{-1, 0, 1\}$.

2205. The force diagram is



The equations are

$$\begin{aligned} \updownarrow : R_2 - R_1 - mg &= 0 \\ \overset{\frown}{R_2} : R_1 - mg &= 0. \end{aligned}$$

Hence, $R_1 = mg$ and $R_2 = 2mg$, which gives the total reaction force as $3mg$ N.

2206. Intersections are at

$$k - x^2 = x^2 \implies x = \pm\sqrt{k/2}.$$

Setting up an integral, the area between the curves is given by

$$\begin{aligned} &\int_{-\sqrt{k/2}}^{\sqrt{k/2}} k - 2x^2 \, dx \\ \equiv &\left[kx - \tfrac{2}{3}x^3\right]_{-\sqrt{k/2}}^{\sqrt{k/2}} \\ \equiv &\tfrac{4k}{3}\sqrt{k/2}. \end{aligned}$$

Equating this to $64\sqrt{3}$,

$$\begin{aligned} &\tfrac{4k}{3}\sqrt{k/2} = 64\sqrt{3} \\ \implies &k^3 = 13824 \\ \implies &k = 24. \end{aligned}$$

2207. Squaring the equations and adding them, we get $(x + y)^2 + (x - y)^2 = 1$. Multiplying out, we get $2x^2 + 2y^2 = 1$, which is a circle centred at $(0, 0)$, with radius $\frac{1}{2}\sqrt{2}$.

2208. The numerator has a factor of $(p - q)$. Having taken it out, we can then simplify and take the limit:

$$\begin{aligned} \frac{dy}{dx} &= \lim_{p,q \to x} \frac{p^3 - q^3}{p - q} \\ &\equiv \lim_{p,q \to x} \frac{(p - q)\big(p^2 + pq + q^2\big)}{p - q} \\ &\equiv \lim_{p,q \to x} p^2 + pq + q^2 \\ &\equiv 3x^2, \text{ as required.} \end{aligned}$$

— Nota Bene —

Starting this question, how can you tell that $(p-q)$ is a factor of the cubic?

- Firstly, by a standard result: a difference of two nth powers always has such a factor.
- Secondly, by the factor theorem, since $p = q$ is a root of the cubic.
- Thirdly, because the question wouldn't work otherwise! It would be impossible to take the limit.

2209. We classify by the longest string of black beads.

① With each black bead alone, there is only one distinct arrangement: BWBWBW.

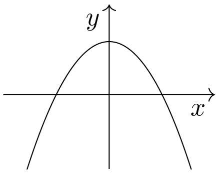
② With two of the black beads together (but not three), we have WBBW**. Filling the last two slots doesn't give distinct arrangements, since WBBWWB and WBBWBW can be rotated and reflected onto each other. So, there is only one distinct arrangement.

③ With all three black beads together, there is only one distinct arrangement: BBBWWW.

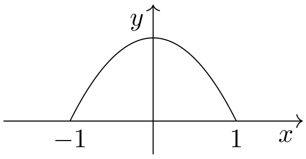
So, there are three arrangements in total.

2210. We could use the chain rule here. However, by the second Pythagorean trig identity, we can see that the expression being differentiated is identical to 1. Since it is constant, its derivative is zero. □

2211. Squaring, we have $y = 1 - x^2$. This is a standard negative parabola, crossing the y axis at 1.



However, the points with negative y values do not satisfy the original equation, as the square root function is undefined for negative inputs. So, the sketch is



2212. Let the short sides of the rectangles have length x and the long sides $^4/_x$. The side lengths of the shaded rhombus are $x\sqrt{2}$. Hence, its area is $x^2\sqrt{2}$, which gives $x = 1$. So, the dimensions are 1×4.

2213. This is a quadratic in b:

$$\begin{aligned} a &= \frac{b^2}{2b-2} \\ \Longrightarrow 2ab - 2a &= b^2 \\ \Longrightarrow b^2 + 2ab - 2a &= 0 \\ \Longrightarrow b &= -a \pm \sqrt{a^2 - 2a}. \end{aligned}$$

2214. The integral of $\sec^2$ is tan. So,

$$\begin{aligned} &\int_0^{\frac{\pi}{6}} \sec^2 2x \, dx \\ &= \left[\tfrac{1}{2}\tan 2x\right]_0^{\frac{\pi}{6}}. \end{aligned}$$

Since $\tan 0 = 0$, this is $\frac{1}{2}\tan\frac{\pi}{3}$, which is $\frac{\sqrt{3}}{2}$.

2215. (a) The average velocity is

$$\begin{aligned} \bar{\mathbf{v}} &= \tfrac{1}{5}\int_0^5 10t - 3t^2 \, dt \\ &= \tfrac{1}{5}\left[5t^2 - t^3\right]_0^5 \\ &= \tfrac{1}{5}\big((125 - 125) - (0)\big) \\ &= 0. \end{aligned}$$

(b) To find the average speed, we first set $v = 0$, which gives $t = 0$ or $t = {}^{10}/_3$. The velocity is positive on $(0, {}^{10}/_3)$ and negative on $({}^{10}/_3, 5]$.

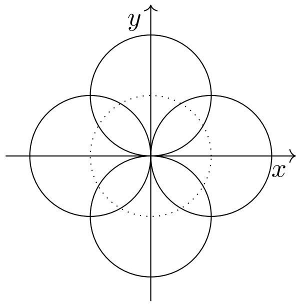
Applying a minus sign to render speed positive in the second period,

$$\begin{aligned} \bar{v} &= \tfrac{1}{5}\left(\int_0^{\frac{10}{3}} 10t - 3t^2 \, dt - \int_{\frac{10}{3}}^5 10t - 3t^2 \, dt\right) \\ &= \tfrac{1}{5}\left(\tfrac{500}{27} + \tfrac{500}{27}\right) \\ &= 7.\dot{4}0\dot{7}. \end{aligned}$$

The average speed is 7.41 ms^{-1} (3sf).

2216. The gradients are reciprocals, which means that the lines are reflections $y = x + c$ for some c. This must go through the intersection of the two lines, which is at $(-\sqrt{3}, 0)$. Substituting this back in, we get $y = x + \sqrt{3}$.

2217. The trigonometric parts dictate the centres of the circles, which lie at the intersections of the unit circle with the coordinate axes:



2218. If the base is blue, then the other four faces must all be yellow (1 outcome). If it is yellow, then the remaining faces can be YYYY (1), YYYB (4), or YBYB (2). The total is 8 outcomes out of 32, so the probability is $\frac{1}{4}$.

2219. The binomial distribution gives

$$\begin{aligned} &\mathbb{P}(X = 1) = \mathbb{P}(X = 2) \\ \Longrightarrow\ &{}^n\mathrm{C}_1 \cdot \tfrac{1}{3} \cdot \tfrac{2}{3}^2 = {}^n\mathrm{C}_2 \cdot \tfrac{1}{3}^2 \cdot \tfrac{2}{3} \\ \Longrightarrow\ &\frac{n!}{1!(n-1)!} \cdot \tfrac{4}{27} = \frac{n!}{2!(n-2)!} \cdot \tfrac{2}{27} \\ \Longrightarrow\ &4n = n(n-1) \\ \Longrightarrow\ &n = 0, 5. \end{aligned}$$

We reject $n = 0$ as it cannot be the number of trials in a binomial distribution. So, $n = 5$.

2220. This is not correct. Friction does act to oppose motion or potential motion. But the friction acts here to oppose the spinning of the wheels, i.e. the backwards motion of the bottom part of the wheel, relative to the road. Hence, the frictional force acts forwards on the wheels, accelerating the car.

— Nota Bene —

Friction acts to oppose the relative motion of two parallel surfaces. Often, if an object has internal moving parts, this is not the same as the absolute motion of an object.

2221. Consider $y = \mathrm{f}(p - x)$ and $y = \mathrm{f}(p + x)$ as graphs. The transformation between these is reflection in the line $x = p$. For any parabola, there is one such line which leaves the graph unchanged, which is the line of symmetry, through the vertex. So, if p is chosen such that $x = p$ is the vertex of $y = \mathrm{f}(x)$, then the required inequality holds. □

2222. We write a as $e^{\ln a}$ and use the chain rule:

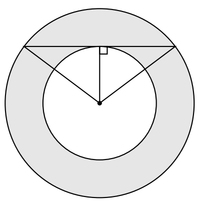
$$\begin{aligned} \frac{d}{dx}\left(e^{\ln a}\right)^x &\equiv \frac{d}{dx}e^{x \ln a} \\ &\equiv \ln a \cdot e^{x \ln a} \\ &\equiv \ln a \cdot a^x. \end{aligned}$$

This proves that $\frac{d}{dx}a^x \equiv \ln a \cdot a^x$.

2223. These graphs have vertical asymptote if and only if their denominators have a root. So, we test the discriminants: these are

(a) $\Delta = -4 < 0$: false,
(b) $\Delta = -3 < 0$: false,
(c) $\Delta = 0$: true.

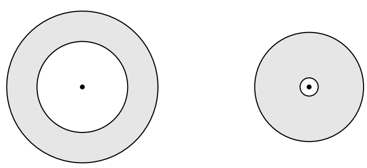
2224. The scenario is



The area is $\pi\left(R^2 - r^2\right) = 25\pi$, so $R^2 - r^2 = 25$. By Pythagoras, then, half the length of the chord d is $d^2 = R^2 - r^2 = 25$. So, $d = 5$ and the chord has length 10.

— Nota Bene —

There is a cunning way of solving this problem, which rests on the fact that the dimensions of the annulus aren't fixed in the question. The annuli shown below have the same area:

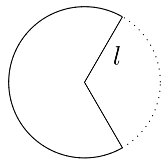


If the question is to make sense, then the answer must be the same whatever the dimensions. In the limiting case, $r = 0$, the annulus is a disc and the chord is a diameter. Since $A = 25\pi$, the radius is 5 and the diameter 10.

2225. The quadratic term must be $(3x - 1)^2$, in order to produce $9x^2$. The by-product is $-6x + 1$. So, the linear term must be $-4(3x - 1)$, in order to produce $-18x$. The constant term must be $+2$:

$$9x^2 - 18x + 7 \equiv (3x - 1)^2 - 4(3x - 1) + 2.$$

2226. The slant height is $l = \sqrt{r^2 + h^2}$. We unwrap the curved surface to form a sector with radius l:



The arc length is the circumference of the base of the cone, i.e. πr^2. The full (dotted) circumference is πl^2. So, the fraction of the circle taken up by the sector is $\frac{2\pi r}{2\pi l} = \frac{r}{l}$. The area, then, is given by

$$\tfrac{r}{l}\pi l^2 = \pi r l = \pi r\sqrt{r^2 + h^2}.$$

Adding the base, which has area πr^2, we get the required result:

$$A = \pi r\left(r + \sqrt{r^2 + h^2}\right).$$

2227. (a) Every hypothesis test tests a claim about an underlying population. The sample is then used to do the testing. The definition of p, which concerns the claim, reflects this fact.

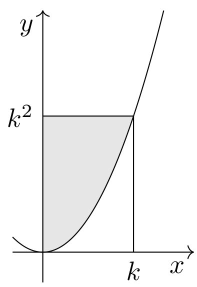
(b) Since we have a two-tail test, the alternative hypothesis is $H_1 : p \neq 0.042$.

(c) Assuming H_0, the number of allergic reactions in the sample has distribution $B(100, 0.042)$. So, $np = 100 \times 0.042 = 4.2$ allergic reactions.

(d) The p-value is the probability that a sample as extreme as this one occurs. In this case, comparing it to the significance level, we have $0.0288 > 0.01$.

There is insufficient evidence, at the 1% level, to reject H_0. It seems the new drug doesn't change the probability of allergic reaction.

2228. The relevant areas are shown below:



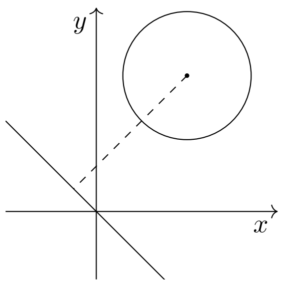
The area of the unshaded region below the curve is given by the usual integral of x^2 between 0 and k. The area of the shaded region to the left of the curve is then given by the second integral, since $\sqrt{y} = x$: rather than integrating y with respect to x, we are integrating x with respect to y. So, the two areas must add to that of the rectangle, which is k^3.

2229. A function is invertible iff it is one-to-one, so a quadratic function is not invertible over an interval which contains points either side of the stationary point. Since this quadratic function is invertible on intervals either side of $x = k$, we know that $x = k$ must be the location of the vertex. Hence, in completed square form, the function must be $\mathrm{f}(x) = a(x-k)^2 + b$, for some constants a, b. □

2230. The hexagon has side length $\sqrt{2}/2$, since each side forms a $(1/2, 1/2, \sqrt{2}/2)$ right-angled triangle with two half-edges of the cube. We can then consider the hexagon as six equilateral triangles, and use $\frac{1}{2}ab\sin C$. This gives the area of the hexagon as

$$\begin{aligned} A_{\text{hex}} &= 6 \times \tfrac{1}{2} \cdot \tfrac{\sqrt{2}}{2} \cdot \tfrac{\sqrt{2}}{2} \cdot \sin 60° \\ &= \tfrac{3\sqrt{3}}{4}. \end{aligned}$$

2231. The shortest distance between line and circle is a normal, which is an extended radius. The circle is $(x-4)^2 + (y-6)^2 = 8$, so the centre is $(4, 6)$ and the radius is $\sqrt{8}$. The sketch is



The dashed line has equation $y = x + 2$, so the intersection is at $(-1, 1)$. The shortest distance, then, is $\sqrt{5^2 + 5^2} - \sqrt{8} = 3\sqrt{2}$, as required.

2232. This is a quadratic is $\sqrt{b}$ (and a^2, in fact) which we can solve by factorising, as follows:

$$\begin{aligned} & a^4 b - 4a^2\sqrt{b} + 4 = 0 \\ \Longrightarrow\ & (a^2\sqrt{b} - 2)^2 = 0 \\ \Longrightarrow\ & a^2\sqrt{b} = 2 \\ \Longrightarrow\ & b = \frac{4}{a^4}. \end{aligned}$$

2233. Since the equation is true whatever the value of a and b, the function $|\mathrm{f}(x)|$ must be identical to the function $\mathrm{f}(x)$. In other words, $\mathrm{f}(x) \geq 0$ for all x. This is proved rigorously below.

Assume, for a contradiction, that $\mathrm{f}(x) < 0$ on an interval (a, b). This gives an interval (a, b) over which the definite integral of f is negative, i.e. the LHS is negative. But the integrand on the RHS is non-negative, so the RHS itself is non-negative. This is a contradiction.

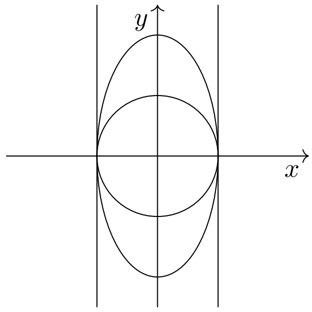
So, since $\mathrm{f}(x)$ is a polynomial (and thus doesn't have any discontinuities), it must be non-negative everywhere.

2234. The graphs can be thought of as transformations of a unit circle $x^2 + y^2 = 1$. The replacement of y by ky corresponds to a stretch scale factor $\frac{1}{k}$ in the y direction. This gives

(a) a unit circle,

(b) a unit circle stretched by scale factor 2 in the y direction, i.e. an ellipse,

(c) a unit circle "stretched" with an infinite scale factor in the y direction, giving two straight lines.



2235. (a) The two objects can be considered as one, since they have the same acceleration. In NII, the internal forces F_2 cancel, leaving the required equation.

(b) NII for the first object is $F_2 - F_1 = m_1 a$, which we rearrange to $a = (F_2 - F_1)/m_1$. Subbing this into the answer from (a),

$$\begin{aligned} & F_3 - F_1 = (m_1 + m_2)\frac{F_2 - F_1}{m_1} \\ \Longrightarrow\ & F_2(m_1 + m_2) = F_1 m_2 + F_3 m_1. \end{aligned}$$

2236. The cross-section is an equilateral triangle of side length l. Using $A_\triangle = \frac{1}{2}ab\sin C$, the cross-sectional area is $\frac{1}{2}l^2 \sin 60°$. Multiplying this by length l,

$$\begin{aligned} A_{\text{prism}} &= \tfrac{1}{2} l^2 \sin 60° \times l \\ &\equiv \tfrac{1}{2} l^2 \tfrac{\sqrt{3}}{2} \times l \\ &\equiv \tfrac{\sqrt{3}}{4} l^3, \text{ as required.} \end{aligned}$$

2237. Using the formula for the sample variance,

$$s_z^2 = \frac{\sum z^2 - 2n\bar{z}^2}{2n}$$

And, since the individual samples have mean zero, we know that $\bar{z} = 0$. Taking out a factor of a half,

$$\begin{aligned} s_z^2 &= \tfrac{1}{2} \frac{\sum z^2}{n} \\ &= \tfrac{1}{2} \left(\frac{\sum x^2}{n} + \frac{\sum y^2}{n} \right) \\ &= \tfrac{1}{2} \left(s_x^2 + s_y^2 \right), \text{ as required.} \end{aligned}$$

2238. Using log rules, we can rewrite both sides over base 3. On the LHS, we get $\log_3 x^2 + \log_3(x+1)$, which is $\log_3 x^2(x+1)$. On the RHS, we can square root both base and argument, giving $\log_9 144 = \log_3 12$. Then, raising 3 to the power of both sides,

$$\begin{aligned} & x^2(x+1) = 12 \\ \Longrightarrow\ & x^3 + x^2 - 12 = 0. \end{aligned}$$

This has a root $x = 2$. Taking the relevant factor out, $(x-2)(x^2+3x+6) = 0$. The quadratic has $\Delta = -15 < 0$, so the solution is $x = 2$.

2239. The GP information yields $b/a = c/b$, i.e. $b^2 = ac$. The AP information yields $b^2 - a^2 = c^2 - b^2$, which is $2b^2 = a^2 + c^2$. Subbing the former into the latter,

$$\begin{aligned} & 2ac = a^2 + c^2 \\ \Longrightarrow\ & (a-c)^2 = 0 \\ \Longrightarrow\ & a = c. \end{aligned}$$

So, $r^2 = 1$, which gives $r = \pm 1$, as required.

2240. We need to express

$$15x^4 - 8x^3 - 12x^2 + 9x + 6$$

in the form

$$(3x+2)(ax^3 + bx^2 + cx + d).$$

The quartic term requires $a = 5$, then the cubic term $b = -6$, the quadratic term $c = 0$, and the constant term $d = 3$. Hence the quotient is $5x^3 - 6x^2 + 3$.

— Alternative Method —

The polynomial long division is

$$\begin{array}{rl} & 5x^3 \quad - 6x^2 + 0x + 3 \\ \hline 3x+2\,) & 15x^4 \quad - 8x^3 - 12x^2 + 9x + 6 \\ & -\,15x^4 - 10x^3 \\ \hline & -\,18x^3 - 12x^2 \\ & 18x^3 + 12x^2 \\ \hline & 0x^2 + 9x \\ & 0x^2 + 0x \\ \hline & 9x + 6 \\ & -\,9x - 6 \\ \hline & 0 \end{array}$$

2241. Making the trig functions the subjects,

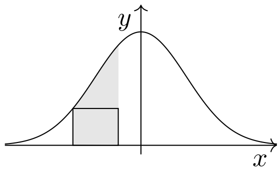
$$\begin{aligned} \sin 2t &= x - 5, \\ \cos 2t &= \tfrac{1}{3}(y-2). \end{aligned}$$

We then square both equations and add them. On the LHS, we get 1 by the first Pythagorean identity. This yields

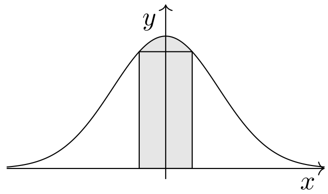
$$1 = (x-5)^2 + \tfrac{1}{9}(y-2)^2.$$

2242. In each case, the normal probability is given by the shaded area.

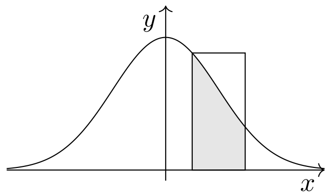
(a) $\mathbb{P}(-1.5 < X < -0.5)$ is underestimated, and the percentage error is large:



(b) $\mathbb{P}(-0.5 < X < 0.5)$ is also underestimated, but the percentage error is significantly lower:



(c) $\mathbb{P}(0.5 < X < 1.5)$ is overestimated, and the percentage error is approximately that in (a):



2243. The gradient of the line is

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{p^2 + 1 - (p+1)}{p^2 - 1 - (p-1)} \\ &\equiv \frac{p^2 - p}{p^2 - p} \\ &\equiv 1. \end{aligned}$$

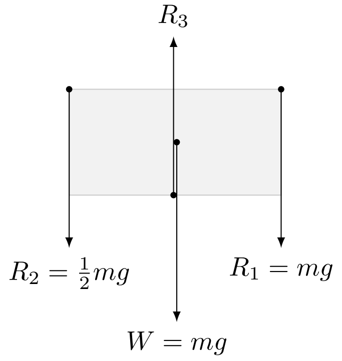
So, using $y - y_1 = m(x - x_1)$, the line has equation

$$\begin{aligned} & y - (p+1) = x - (p-1) \\ \Longleftrightarrow\ & y = x + 2. \end{aligned}$$

2244. The bricks on the middle storey are unstable. We will show this for the middle-right brick, which we label brick B. Acting on brick B, there is weight $W = mg$, and three normal reaction forces. Analysing these:

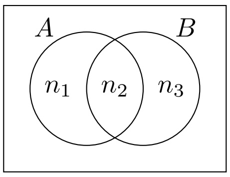
- The top-right brick is on the point of toppling outwards. So, its reaction R_1, magnitude mg, acts at the right-hand edge of B.
- The pile is symmetrical horizontally, so the greatest moment the top-centre brick could apply to B would be with a reaction R_2 of magnitude $\frac{1}{2}mg$ at the left-hand edge of B.
- The reaction force R_3 from the bottom brick cannot prevent B from toppling.

Hence, the least possible moment clockwise on B occurs in the following configuration of forces:



The clockwise moment around the centre of mass is $mgl - \frac{1}{2}mgl > 0$. Hence, the middle bricks must topple outwards.

2245. Label the numbers of elements as below:



Then the required formula is simply the result that $n_1 + n_2 + n_3 \equiv (n_1 + n_2) + (n_2 + n_3) - n_2$.

——— Nota Bene ———

This is a discrete version of the inclusion-exclusion principle for probabilities:

$$\mathbb{P}(A \cup B) \equiv \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

2246. We enact the differential operator $\frac{d}{dx}$, using the product rule. This gives

$$\begin{aligned} &\frac{d}{dx}(xy) = 1 \\ \Longrightarrow\ &1 \cdot y + x \cdot \frac{dy}{dx} = 1 \\ \Longrightarrow\ &\frac{dy}{dx} = \frac{1-y}{x}. \end{aligned}$$

2247. Rearranging to $x = y - 1$, we can substitute and multiply out by the binomial expansion, giving

$$\begin{aligned} &x^3 + 3x^2 + 3x + 6 \\ \equiv\ &(y-1)^3 + 3(y-1)^2 + 3(y-1) + 6 \\ \equiv\ &y^3 + 5. \end{aligned}$$

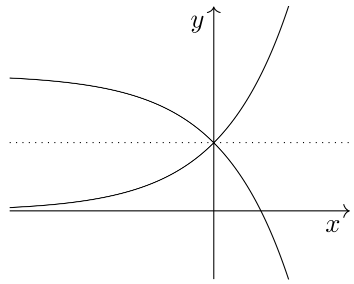
——— Alternative Method ———

We need $(x+1)^3$ to produce the x^3 term. By the binomial expansion, this gives $x^3 + 3x^2 + 3x + 1$. This matches the x^2 and x terms, which means we only need to add 5. This gives $y^3 + 5$.

2248. The areas are $A_i = \pi i^2 - \pi(i-1)^2 = (2i-1)\pi$. The total area is 25π, so the probabilities are $\frac{2i-1}{25}$. The expectation is

$$\begin{aligned} \mathbb{E}(S) &= \sum_{i=1}^{5} \frac{(6-i)(2i-1)}{25} \\ &= \tfrac{1}{25}(5 \cdot 1 + 4 \cdot 3 + 3 \cdot 5 + 2 \cdot 7 + 1 \cdot 9) \\ &= 2.2, \text{ as required.} \end{aligned}$$

2249. The two transformations are reflection in the x axis followed by translation by vector $2\mathbf{j}$. The combined effect of these two may be described as reflection in the line $y = 1$.



2250. Differentiating by the product rule,

$$\begin{aligned} &y = (x+1)^3 \left(\tfrac{1}{2}x^2 + x + 1\right) \\ \Longrightarrow\ &\frac{dy}{dx} = 3(x+1)^2 \left(\tfrac{1}{2}x^2 + x + 1\right) \\ &\qquad\qquad + (x+1)^3(2x+1). \end{aligned}$$

Taking out a common factor of $(x+1)^2$, this can be simplified to

$$\frac{dy}{dx} = (x+1)^2 \left(\tfrac{5}{2}x^2 + 5x + 4\right).$$

We can also simplify as follows:

$$\frac{3y}{x+1} = (x+1)^2 \left(\tfrac{3}{2}x^2 + 3x + 3\right).$$

Subtracting these two expressions and taking out a factor of $(x+1)^2$, the LHS is

$$\begin{aligned} &\frac{dy}{dx} - \frac{3y}{x+1} \\ &= (x+1)^2(x^2 + 2x + 1) \\ &\equiv (x+1)^4. \end{aligned}$$

This is the RHS, so the implication holds.

2251. Expanding $R\cos(\theta - \alpha)$ with a compound-angle formula, we want to find R and α such that

$$\begin{aligned} &R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \\ \equiv\ &\sqrt{3}\cos\theta + \sin\theta. \end{aligned}$$

Equating coefficients,

$$\begin{aligned}\cos\theta &: R\cos\alpha = \sqrt{3}\\ \sin\theta &: R\sin\alpha = 1.\end{aligned}$$

Squaring and adding the equations gives $R = 2$. Dividing the equations gives $\tan\alpha = \frac{1}{\sqrt{3}}$, thus $\theta = \frac{\pi}{6}$. Substituting back in,

$$\begin{aligned}&2\cos\left(\theta - \tfrac{\pi}{6}\right) = 1\\ \Longrightarrow\ &\theta - \tfrac{\pi}{6} = -\tfrac{\pi}{3}, \tfrac{\pi}{3}\\ \Longrightarrow\ &\theta = -\tfrac{\pi}{6}, \tfrac{\pi}{2}.\end{aligned}$$

2252. Using the quotient rule,

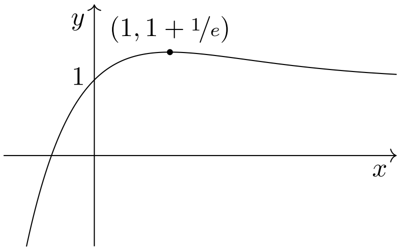
$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x+1)e^x - (e^x+x)e^x}{e^{2x}}\\ &\equiv \frac{e^x + xe^x}{e^{2x}}\\ &\equiv \frac{1+x}{e^x}.\end{aligned}$$

Setting this to zero, $x = -1$ is a stationary point. Differentiating again by the quotient rule,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{e^x - (1+x)e^x}{e^{2x}}\\ &\equiv \frac{x}{e^x}.\end{aligned}$$

Evaluating at $x = -1$, the second derivative is $-e < 0$, so the SP is a local maximum.

To sketch, we rearrange to $y = 1 + xe^{-x}$, which shows us that, as $x \to \infty$, $y \to 1$, and as $x \to -\infty$, $y \to -\infty$. Putting this information together, the graph is



2253. We know that $x^2 \equiv |x|^2$, because squares and mods are both positive:

$$\begin{aligned}&6x^2 + |x| = 1\\ \Longrightarrow\ &6|x|^2 + |x| - 1 = 0\\ \Longrightarrow\ &(3|x|-1)(2|x|+1) = 0\\ \Longrightarrow\ &|x| = \tfrac{1}{3}, -\tfrac{1}{2}.\end{aligned}$$

We reject the latter option, as $|x|$ must be positive, which gives $x = \pm\frac{1}{3}$.

2254. These two are equally likely. Explicitly,

$$\tfrac{13}{52} \times \tfrac{12}{51} \times \tfrac{13}{50} = \tfrac{13}{52} \times \tfrac{13}{51} \times \tfrac{12}{50}.$$

Multiplication is commutative ($12 \times 13 = 13 \times 12$), so the order in which the cards are dealt doesn't affect the probability.

2255. (a) This is not valid: a counterexample is $a = 0$, $b = -1$.

(b) This is valid: the function $x \mapsto x^3$ is always increasing (or else stationary at $x = 0$), which means it preserves the direction of inequality.

(c) This is not valid: $a = 0$, $b = -1$ disproves it.

2256. Call the integers $2k-1$, $2k+1$ and $2k+3$.

Suppose neither $2k-1$ nor $2k+1$ is divisible by 3. Since every third integer is divisible by 3, this means that $2k$ is divisible by 3. This implies that $2k+3$ is divisible by 3. Therefore, at least one of the integers must be divisible by 3. Hence, their product is divisible by 3. □

2257. Using the fact that $\log_a b \equiv \log_{a^n} b^n$, we rewrite all terms over base 2. This is

$$\log_2 x + \log_2 x^{\frac{1}{2}} + \log_2 x^{\frac{1}{3}} + ...$$

We combine these with a log rule, giving $\log_2 x^k$, where $k = 1 + \frac{1}{2} + \frac{1}{3} + ...$ We are told that the index k diverges. So, since $x > 1$, x^k must also diverge. Hence, $\log_2 x^k$ must also diverge. □

2258. (a) This is linear with a positive gradient, so there is a strong positive correlation.

(b) This is a parabolic relationship centred at $(0,0)$, with $x = 0$ as a line of symmetry. The line of best fit will be approximately the x axis, giving approximately zero correlation.

(c) As in (a), strong negative correlation.

(d) The relationship is not linear, but it is cubic, which, being odd, has (very) broadly the same shape as a linear relationship. The cubic is positive, so weak positive correlation.

(e) As in (d), weak negative correlation.

2259. (a) The first derivative is $\frac{dy}{dx} = 6x^2 - 4x^3$. We set this to 2, which gives $2x^3 - 3x^2 + 1 = 0$. Solving using a calculator, the gradient is 2 at $(1,1)$ and $(-1/2, -5/16)$. The first of these lies on $y = 2x - 1$, so the line $y = 2x - 1$ is tangent to the curve at $(1,1)$.

(b) The second derivative is $12x(1-x)$. This is zero at $x = 1$ and changes sign (single factor of $(1-x)$), so $(1,1)$ is a point of inflection.

(c) We solve $2x^3 - x^4 = 2x - 1$ to find the other intersection: it's at $x = -1$. We then calculate the enclosed area with the following integral:

$$\begin{aligned}A &= \int_{-1}^{1} (2x^3 - x^4) - (2x-1)\,dx\\ &= \left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5 - x^2 + x\right]_{-1}^{1}\\ &= \left(\tfrac{1}{2} - \tfrac{1}{5} - 1 + 1\right) - \left(\tfrac{1}{2} + \tfrac{1}{5} - 1 - 1\right)\\ &= \tfrac{8}{5}, \text{ as required.}\end{aligned}$$

2260. The sum of the interior angles of a polygon is $(n-2)\pi$. Hence, the mean angle is $\frac{n-2}{n}\pi$. So, the required limit is

$$\begin{aligned}\bar{\theta} &= \lim_{n\to\infty} \frac{n-2}{n}\pi \\ &= \lim_{n\to\infty} \frac{1-\frac{2}{n}}{1}\pi \\ &= \pi.\end{aligned}$$

2261. This is a quadratic in $\sqrt{x}$. The formula gives

$$\begin{aligned} & px + q\sqrt{x} + r = 0 \\ \Longrightarrow\ & \sqrt{x} = \frac{-q \pm \sqrt{q^2-4pr}}{2p} \\ \Longrightarrow\ & x = \left(\frac{-q \pm \sqrt{q^2-4pr}}{2p}\right)^2. \end{aligned}$$

2262. The two geometric series have nth terms $1/2^n$ and $(-1/2)^n$. So, their first terms, which are also their common ratios, are $1/2$ and $-1/2$. The individual sums to infinity, using $\frac{a}{1-r}$, are 1 and $-1/3$. Since both sums have $|r| < 1$ and thus converge, so must their sum. This gives $1 - 1/3 = 2/3$.

2263. A curve is inflected at a point of inflection, where the second derivative is zero and changes sign. When we differentiate twice, the linear part goes:

$$\begin{aligned} & y = \mathrm{f}(x) + ax + b \\ \Longrightarrow\ & \frac{dy}{dx} = \mathrm{f}'(x) + a \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = \mathrm{f}''(x). \end{aligned}$$

The curves have the same second derivative. So, they must have exactly the same set of points of inflection. QED.

2264. (a) AB has gradient $\frac{1}{7}$ and midpoint $(6.5, 0.5)$. Hence, the line $y = 54 - 7x$ is the perpendicular bisector of AB. Since the centre is equidistant from A and B, it must lie on this line.

(b) The expressions for squared distance are

i. $(54-7p)^2$,

ii. $(p-3)^2 + (46-7p)^2$.

(c) Equating the squared distances, we have

$$\begin{aligned} & (54-7p)^2 = (p-3)^2 + (46-7p)^2 \\ \Longrightarrow\ & p^2 + 106p - 791 = 0 \\ \Longrightarrow\ & p = 7, -113. \end{aligned}$$

Therefore, possible coordinates for the centre are $(7,5)$ or $(-113, 845)$.

2265. The LHS is

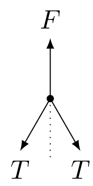
$$\begin{aligned} & T_n + T_{n-1} \\ &= \tfrac{1}{2}n(n+1) + \tfrac{1}{2}(n-1)n \\ &\equiv \tfrac{1}{2}n\big((n+1)+(n-1)\big) \\ &\equiv \tfrac{1}{2}n(2n) \\ &\equiv n^2. \end{aligned}$$

The RHS is

$$\begin{aligned} & (T_n - T_{n-1})^2 \\ &= \left(\tfrac{1}{2}n(n+1) - \tfrac{1}{2}(n-1)n\right)^2 \\ &\equiv \left(\tfrac{1}{2}n\big((n+1)-(n-1)\big)\right)^2 \\ &\equiv \left(\tfrac{1}{2}n \times 2\right)^2 \\ &\equiv n^2. \end{aligned}$$

Hence, the identity holds.

2266. Since the string is smooth, the tension is the same throughout the loop. The force diagram, then, for the uppermost vertex of the loop is:



Vertically, $F - 2T\cos 30° = 0$. So, $T = \frac{\sqrt{3}}{3}F$.

2267. Differentiating implicitly by the chain rule,

$$\begin{aligned} & \frac{d}{dt}(x+y)^2 \\ &\equiv 2(x+y)\frac{d}{dt}(x+y) \\ &\equiv 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \\ &= 2(x+y)(a+b). \end{aligned}$$

2268. Carrying out the integrals, we combine the two constants of integration into a single $+c$ on the RHS:

$$e^x = e^t + c.$$

Substituting $t = 0$, $x = \ln 2$, we get $2 = 1 + c$, so $c = 1$. The solution curve is therefore $e^x = e^t + 1$. Substituting $t = 3\ln 2 = \ln 9$ gives $x = \ln 10$.

2269. The mean of the interior angles of a pentagon is $3\pi/5$. An AP proceeds linearly, so the third of the five angles must have this value. The smallest and greatest angles are then equidistant from the mean. The lower limit 0 is closer to $3\pi/5$ than the upper limit 2π is.

- Hence, 0 is the lower limit on the smallest angle. It cannot be attained.

- The upper limit, which can be attained, is the regular pentagon, in which all interior angles are $3\pi/5$.

In set notation, therefore, $\theta \in (0, 3\pi/5]$ radians.

2270. Since the curve is symmetrical in the y axis, the double tangent must be parallel to the x axis. To find it, we set the derivative to zero: $\frac{2}{5}x^3 - 2x = 0$, which gives $x = 0, \pm\sqrt{5}$. So, the dashed line has equation $y = -\frac{5}{2}$. The shaded area is given by

$$\begin{aligned}&\int_{-\sqrt{5}}^{\sqrt{5}} \tfrac{1}{10}x^4 - x^2 + \tfrac{5}{2}\,dx\\ &= \left[\tfrac{1}{50}x^5 - \tfrac{1}{3}x^3 + \tfrac{5}{2}x\right]_{-\sqrt{5}}^{\sqrt{5}}\\ &= 2\left(\tfrac{1}{2}\sqrt{5} - \tfrac{5}{3}\sqrt{5} + \tfrac{5}{2}\sqrt{5}\right)\\ &= \tfrac{8}{3}\sqrt{5}, \text{ as required.}\end{aligned}$$

2271. By the chain rule,

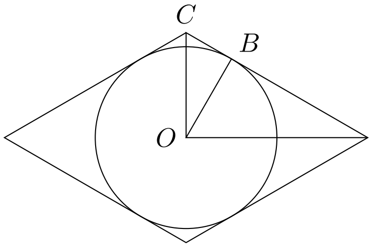
$$\begin{aligned}\frac{dy}{dx} &= \tfrac{1}{2}(\cos 3x)^{-\frac{1}{2}} \cdot \frac{d}{dx}(\cos 3x)\\ &\equiv \tfrac{1}{2}(\cos 3x)^{-\frac{1}{2}} \cdot -\sin 3x \cdot 3\\ &\equiv -\frac{3\sin 3x}{2\sqrt{\cos 3x}}.\end{aligned}$$

2272. Call the output y. We then rearrange to make the input x the subject:

$$\begin{aligned}y &= \frac{2x+1}{2x-1}\\ \Longrightarrow 2xy - y &= 2x + 1\\ \Longrightarrow 2xy - 2x &= y + 1\\ \Longrightarrow x &= \frac{y+1}{2y-2}.\end{aligned}$$

So, the inverse rule is $\mathrm{h}^{-1} : x \mapsto \dfrac{x+1}{2x-2}$.

2273. The situation is as follows:



Let $\alpha = \angle OAB$. Then $|OA| = l\cos\alpha$, and hence $|OB| = l\cos\alpha\sin\alpha$. By the double-angle formula $\sin 2x \equiv 2\sin x\cos x$, this is $|OB| = \frac{1}{2}l\sin 2\alpha$, where 2α is the interior angle at A. This vertex was chosen arbitrarily, so the radius is given by $r = \frac{1}{2}l\sin\phi$, where ϕ is any interior angle.

2274. The expansion is

$$\begin{aligned}&\left(\sqrt{3}\right)^5 + 5\left(\sqrt{3}\right)^4(-2) + 10\left(\sqrt{3}\right)^3(-2)^2\\ &\quad + 10\left(\sqrt{3}\right)^2(-2)^3 + 5\left(\sqrt{3}\right)(-2)^4 + (-2)^5\\ &= 209\sqrt{3} - 362.\end{aligned}$$

The magnitude of $\sqrt{3} - 2$ is 0.26.... When raised to the power five, this is around 0.001, which we can approximate to zero. Hence,

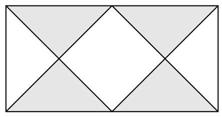
$$209\sqrt{3} - 362 \approx 0.$$

This gives $\sqrt{3} \approx \frac{362}{209}$, as required.

2275. (a) This is true. If $x\sec x = 0$, then one of x and $\sec x$ must be zero. But $\sec x = \frac{1}{\cos x}$ is never zero. We don't need to check the domain of $\sec x$, since the statement $x\sec x = 0$ can only occur where $\sec x$ is well defined.

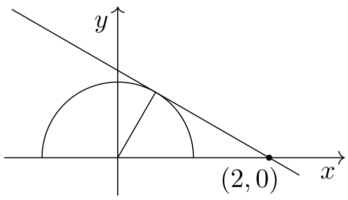
(b) This is not true. A counterexample is $x = \frac{1}{2}\pi$, where $\cot x = 0$. Unlike $\sec x$, $\cot x$ has roots.

2276. There are ${}^7\mathrm{C}_4 = 35$ ways of shading four regions. Of these, there is only one outcome which has no two shaded regions sharing a border:



Hence, the probability is $\frac{1}{35}$.

2277. The curve is a unit semicircle. So, the scenario is



Since the radius is 1, the angle at $(2, 0)$ is 30°, and the gradient of the tangent is therefore

$$\begin{aligned}m &= -\tan 30°\\ &= -\tfrac{1}{\sqrt{3}}.\end{aligned}$$

Using $y - y_1 = m(x - x_1)$, the equation of the tangent is $\sqrt{3}y + x = 2$.

2278. The mean is $\bar{x} = \frac{-160}{n}$. The variance is

$$s^2 = \frac{\sum x^2 - n\bar{x}^2}{n} = \frac{1228 - \frac{(-160)^2}{n}}{n}.$$

Equating to 1.2024 and multiplying by n^2,

$$\begin{aligned}&1.2024n^2 - 1228n + 25600 = 0\\ \Longrightarrow\ &n = 29.29, 1000.\end{aligned}$$

Since the values given were exact, we reject the non-integer solution. So, $n = 1000$.

2279. Call the function f. Consider the graph $y = \mathrm{f}(x)$.

The second derivative $\mathrm{f}''(x)$ is positive for all x. So, the first derivative $\mathrm{f}'(x)$ is increasing for all x. Furthermore, $\mathrm{f}''(x)$ must have even degree, so $\mathrm{f}'(x)$ must have odd degree. Hence, $\mathrm{f}'(x) = 0$ has exactly one root. The curve therefore has exactly one SP.

The curve is polynomial, so it can have at most one x intercept either side of its SP. So, the function f can have at most two roots. QED.

2280. Since the vertices are on the coordinate axes, the triangle is right-angled. So, whatever changes a and b undergo, Pythagoras's theorem still holds. We differentiate it implicitly with respect to time:

$$\begin{aligned} & a^2 + b^2 = c^2 \\ \Longrightarrow\ & \frac{d}{dt}(a^2) + \frac{d}{dt}(b^2) = \frac{d}{dt}(c^2) \\ \Longrightarrow\ & 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt} \\ \Longrightarrow\ & a\frac{da}{dt} + b\frac{db}{dt} = c\frac{dc}{dt}, \text{ as required.} \end{aligned}$$

2281. (a) The standard deviation is positive, so the range is $\mathbb{R}^+$, also known as $[0, \infty)$.

(b) It is invertible. A sample of size 2 with mean zero must be of the form $\{-x, x\}$. Since s is a measure of spread, values of x will be in one-to-one correspondence with values of s. Hence, the function $s = \mathrm{f}(x)$ is invertible.

2282. By the chain rule, the first derivative is

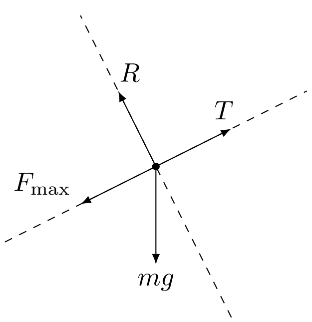
$$\frac{dy}{dx} = 3(e^x - 1)^2 e^x.$$

We set this to zero for SPs. Since $e^x > 0$, this gives $e^x = 1$, so $x = 0$. By the product and chain rules, the second derivative is

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6(e^x - 1)e^{2x} + 3(e^x - 1)^2 e^x \\ &\equiv 3(e^x - 1)e^x(3e^x - 1). \end{aligned}$$

Since it has a single factor of $(e^x - 1)$, this is zero at $x = 0$, and changes sign. Hence, the origin is a stationary point of inflection.

2283. The force diagram is



Acceleration is zero, since the case is being winched at constant speed, and friction is at $F_{\max} = \mu R$, since it is in motion. Perpendicular to the slope, $R = mg\cos\theta$, so $F_{\max} = \mu mg\cos\theta$. Parallel to the slope, $T - mg\sin\theta - F_{\max} = 0$, so

$$T = mg(\sin\theta + \mu\cos\theta).$$

2284. By the reverse chain rule,

$$\begin{aligned} & \int_6^7 \frac{4}{5 - x}\,dx \\ &= -4\Big[\ln|5 - x|\Big]_6^7 \\ &= -4\big(\ln|-2| - \ln|-1|\big) \\ &= -4\ln 2 \\ &= \ln\tfrac{1}{16}. \end{aligned}$$

2285. Differentiating, $\frac{dx}{du} = -2\sin 2u$. Reciprocating,

$$\frac{du}{dx} = \frac{-1}{2\sin 2u}.$$

We can now simplify:

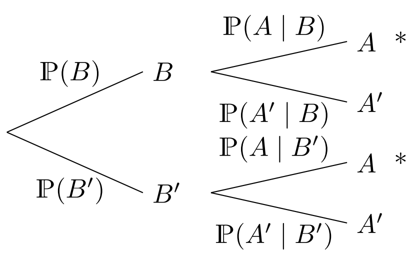
$$\begin{aligned} & 4\frac{du}{dx} + \sec u \operatorname{cosec} u \\ &= \frac{-4}{2\sin 2u} + \frac{1}{\sin u\cos u} \\ &\equiv \frac{-4}{4\sin u\cos u} + \frac{1}{\sin u\cos u} \\ &\equiv 0, \text{ as required.} \end{aligned}$$

2286. The first derivative is $\frac{dy}{dx} = 4x\left(x^2 - 1\right)$, which is zero at $x = 0, \pm 1$. Since the quartic is positive, these three SPs must be two local minima and a local maximum between them.

— Alternative Method —

Alternatively, the curve is $y = (x - 1)^2(x + 1)^2$, which has double roots at $x = \pm 1$. These are two local minima. And, since $y \geq 0$ everywhere, there must be a local maximum between them.

2287. A tree diagram, conditioned on B, is



The formula calculates the probability of B given A, which requires restricting the possibility space to the two branches marked $*$. The formula is then the upper one (B) out of both of them.

2288. (a) The points are
   i. P
   ii. Q
   iii. The midpoint of PQ.

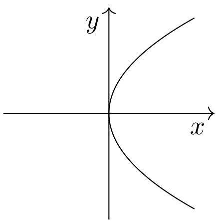
(b) The point is at $\lambda = \frac{5}{8}$, which gives

$$\begin{pmatrix} a \\ b \end{pmatrix} + \frac{5}{8}\begin{pmatrix} c-a \\ d-b \end{pmatrix} \equiv \frac{1}{8}\begin{pmatrix} 3a+5c \\ 3b+5d \end{pmatrix}.$$

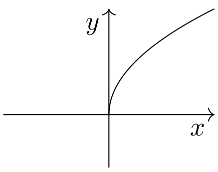
2289. Exponentiating both sides,

$$\begin{aligned} x &= e^{1+2\ln y} \\ &\equiv e^{1+\ln y^2} \\ &\equiv e \times e^{\ln y^2} \\ &\equiv ey^2. \end{aligned}$$

Since e is a constant, this is a parabola with vertex at the origin:



However, the original equation is undefined for negative y, since logarithms cannot take negative inputs. So, the graph $\ln x = 1 + 2\ln y$ is restricted to the positive quadrant:



2290. Since $y = 0$ is a diameter of the unit circle, the triangle is right-angled. $2x + y = 2$ passes through $(1, 0)$ and has gradient $m = -2$. Hence, we require the line through $(-1, 0)$ which has gradient $m = \frac{1}{2}$. This is $y = \frac{1}{2}x + \frac{1}{2}$.

2291. By the double-angle formula $\cos 2x \equiv 2\cos^2 x - 1$, we get $\cos^2 x \equiv \frac{1}{2}(\cos 2x + 1)$. Using this,

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \cos^2 x \, dx \\ = \ &\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x + 1 \, dx. \end{aligned}$$

The reverse chain rule gives

$$\begin{aligned} &\tfrac{1}{2}\left[\tfrac{1}{2}\sin 2x + x\right]_0^{\frac{\pi}{2}} \\ = \ &\tfrac{1}{2}\left(\tfrac{1}{2}\sin\pi + \tfrac{\pi}{2}\right) - (0) \\ = \ &\tfrac{\pi}{4}, \text{ as required.} \end{aligned}$$

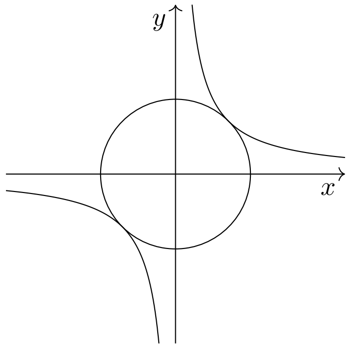
2292. The conditional probability formula gives

$$\begin{aligned} \mathbb{P}(2\text{ H} \mid 1 \text{ or } 2\text{ H}) &= \frac{\mathbb{P}(2\text{ H})}{\mathbb{P}(1 \text{ or } 2\text{ H})} \\ &= \frac{\frac{13}{52} \times \frac{12}{51}}{\frac{13}{52} \times \frac{12}{51} + 2 \times \frac{13}{52} \times \frac{39}{51}} \\ &= \tfrac{2}{15}. \end{aligned}$$

2293. Substituting the equation of the hyperbola into that of the circle,

$$\begin{aligned} & x^2 + \frac{1}{x^2} = 2 \\ \Longrightarrow \ & x^4 - 2x^2 + 1 = 0 \\ \Longrightarrow \ & (x^2 - 1)^2 = 0 \\ \Longrightarrow \ & (x+1)^2(x-1)^2 = 0. \end{aligned}$$

Each root $x = \pm 1$ is a double root. Hence, each is a point of tangency. The graphs are



2294. The second Pythagorean trig identity is

$$1 + \tan^2 x \equiv \sec^2 x.$$

Substituting gives a quadratic in $\tan x$:

$$\begin{aligned} & \sec^2 x - \tan x - 1 = 0 \\ \Longrightarrow \ & \tan^2 x - \tan x = 0 \\ \Longrightarrow \ & \tan x(\tan x - 1) = 0 \\ \Longrightarrow \ & \tan x = 0, 1 \\ \Longrightarrow \ & x = 0, \tfrac{\pi}{4}, \pi, \tfrac{5\pi}{4}. \end{aligned}$$

2295. The derivative is m. So, substituting for y and $\frac{dy}{dx}$, we require $m \equiv x + mx + c + 1$. Since this is an identity, we can equate coefficients: those of x give $m = -1$, then the constant terms give $-1 = c + 1$. Hence, the line that satisfies the DE is $y = -x - 2$.

2296. If the m and $3m$ fireworks form a vertical line, then the $2m$ firework is level horizontally with O. The lines of action of the weights, then, are horizontally r from O, in the case of the $2m$ firework, and $\frac{1}{2}r$ from O, in the case of the other two. Hence, the anticlockwise moment is $2m \times r$ and the clockwise moment is $(m + 3m) \times \frac{1}{2}r$. These cancel out, so the wheel is in equilibrium.

2297. The binomial expansion gives

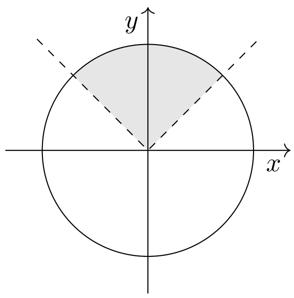
$$\begin{aligned}&\left(\sqrt{x} \pm \sqrt{x-1}\right)^3\\ \equiv\ & x^{\frac{3}{2}} \pm 3x(x-1)^{\frac{1}{2}} + 3x^{\frac{1}{2}}(x-1) \pm (x-1)^{\frac{3}{2}}.\end{aligned}$$

When we subtract, the first and third terms cancel:

$$\begin{aligned}& 6x(x-1)^{\frac{1}{2}} + 3(x-1)^{\frac{3}{2}} = 0\\ \Longrightarrow\ & (x-1)^{\frac{1}{2}}(2x + (x-1)) = 0\\ \Longrightarrow\ & (x-1)^{\frac{1}{2}}(3x-1) = 0\\ \Longrightarrow\ & x = 1, \tfrac{1}{3}.\end{aligned}$$

However, we reject the latter root, as the square root of $-\frac{2}{3}$ is undefined. So, the solution is $x = 1$.

2298. We require the region within a circle of radius 2, centred at the origin, and strictly above the mod graph $y = |x|$:



2299. The relevant compound-angle formula is

$$\sin(x \pm y) \equiv \sin x \cos y \pm \cos x \sin y.$$

When we substitute these, the $\sin x \cos y$ terms cancel, leaving

$$\begin{aligned}& \frac{dy}{dx} = -2\cos x \sin y\\ \Longrightarrow\ & \operatorname{cosec} y \frac{dy}{dx} = -2\cos x.\end{aligned}$$

2300. The -1 makes no difference here; it doesn't affect the gradient. For n odd, there are no values of x for which $f(x)$ is decreasing; for n even, $f(x)$ is decreasing for negative x. So, the probability is $\frac{2}{5}$.

——— End of 23rd Hundred ———